**Environmental Resources Research**
Vol. 8, No. 1, 2020

GUASNR

# Evaluation of estimation methods for parameters of the probability functions in tree diameter distribution modeling

**M. Teimouri*[1], K. Abdolahnezhad[2], Sh. Ghalandarayeshi[1]**
[1]Assistant Professor, Gonbad Kavous University, Gonbad Kavous, Iran
[2]Assistant Professor, Golestan University, Gorgan, Iran

*Received: May 2019 ; Accepted: January 2020*

## Abstract
One of the most commonly used statistical models for characterizing the variations of tree diameter at breast height is Weibull distribution. The usual approach for estimating parameters of a statistical model is the maximum likelihood estimation (likelihood method). Usually, this works based on iterative algorithms such as Newton-Raphson. However, the efficiency of the likelihood method is not guaranteed since there is no assurance that the Newton-Raphson method for maximizing the log-likelihood function will converge. In such cases, one option is to use a better estimation approach. In this study, several methods were compared for estimating the parameters of two- and three-parameter Weibull distributions. We applied ten methods for two-parameter and twelve methods for three-parameter cases. The data set was collected from natural beech dominated forest in northern Iran. The results demonstrated that among the estimators investigated for two-parameter Weibull distribution, the percentile method outperformed other competitors. In contrast, for three-parameter Weibull distribution, the trimmed *L*-moment (*TL*-moment) method and the modified method of moments (type I and type II) outperformed other competitors in terms of Cramer Von-Mises criterion and Kolmogorov-Smirnov criterion, respectively.

***Keywords:*** Least square method, Method of maximum product spacings, *TL*-moment, Weibull distribution, Weighted maximum likelihood estimator.

*Corresponding author; teimouri@aut.ac.ir

## Introduction

In applied forestry, diameter distribution is of great importance in describing forest stand structure. It is useful for estimating growth and yield and planning forest management activities (Hyink and Moser, 1983; Burkhart and Tomé, 2012). It can also be used to infer past disturbance event and the developmental stage of the stand (Coomes and Allen, 2007; Ghalandarayeshi et al., 2017).

During the last four decades, numerous probability density functions (pdf) such as gamma, Johnson's SB, lognormal, and Weibull distributions have been examined for modelling distribution of the diameter at breast height (dbh) (Reynolds et al., 1988; Ghalandarayeshi, 2019). Initially, the Weibull distribution was used for explaining the diameter distributions in forestry (Bailey and Dell, 1973). The Weibull model is one of the most commonly used statistical distributions used for modelling dbh. For a review, we refer reader to Bailey and Dell, (1973) for quantifying the variations of the dbh with Weibull distribution, Little, (1983) for modelling dbh of mixed stands through the three-parameter Weibull distribution, Kilkki et al., (1989) for the use of Weibull distribution in modelling dbh of Finnish pine (*Pinus sylvestris*)), Zhang et al., (2003) for comparison of estimation methods for fitting Weibull and Johnson's SB distributions fitted to mixed spruce, Newton et al., (2005) for modelling dbh of black spruce (*Picea mariana* Mill) plantations using three-parameter Weibull distribution, Merganič and Sterba, (2006) for characterizing dbh using Weibull distribution, Zhang and Liu, (2006) for modellig dbh of irregular forest stands by Weibull and modified Weibull distributions, Gorgoso et al., (2007) for modelling dbh of *Betula alba* L. stands using two-parameter Weibull distribution, Lei, (2008) for comarison of three estimators of the Weibull parameters fitted to Chinese pine (*Pinus tabulaeformis*), Stankova and Zlatanov, (2010) for modeling dbh of Austrian black pine plantations based on percentile and projection methods, Zhang et al., (2010) for compatibility of stand basal area predictions based on forecast combination, and Duan et al., (2013) for modelling and estimating dbh using Richards model and three-parameter Weibull distribution. The pdf and cumulative distribution function (cdf) of a three-parameter Weibull distribution are, respectively, given by Norman et al., (1994); Prabhakar Murthy et al., (2004) and Dodson, (2006) as follows:

$$f(x;\alpha,\beta,\theta)=\frac{\alpha}{\beta}\left(\frac{x-\theta}{\beta}\right)^{\alpha-1}\exp\left\{-\left(\frac{x-\theta}{\beta}\right)^{\alpha}\right\}, \quad (1)$$

and

$$F(x;\alpha,\beta,\theta)=1-\exp\left\{-\left(\frac{x-\theta}{\beta}\right)^{\alpha}\right\}, \quad (2)$$

for $x>\theta$, $\alpha>0$, $\beta>0$ and $\theta\in\mathbb{R}$. Here, the parameters $\alpha$, $\beta$ and $\theta$ are known as the shape, scale and location parameters, respectively. If we set $\theta=0$ in Abdul-Moniem, (2007) and Bailey and Dell, (1973), we have the pdf and cdf of a two-parameter Weibull distribution, respectively.

Statistical inference for Weibull distribution parameters has a long history. For two-parameter Weibull distribution we refer to method of L-moment (Hosking, 1990), method of percentile (Hassanein, 1971; Wang and Keats, 1995), weighted least square (Van Zyl and Schall, 2012; Hung, 2001; Zhang et al., 2008; Kantar, 2015), generalized least square (Engeman and Keefe, 1982; Kantar, 2015), weighted maximum likelihood (Jacquelin, 1993), method of moment and logarithmic moment (Wayne, 1982; Norman et al., 1994; Dodson, 2006), and method of rank (Teimouri and Nadarajah, 2012). In the case of three-parameter Weibull distribution, the most popular approach, i.e., the method of maximum likelihood may break down under some situations. To tackle this issue, the method of modified maximum likelihood and modified moments have been proposed (Cohen and Whitten, 1982). Also, weighted maximum likelihood estimators have been introduced in the literature (Cousineau, 2009). There are other methods developed in the literature for estimating the parameters of three-parameter Weibull distribution, among them we refer to *TL*-moment (Teimouri et al., 2013), and method of maximum product spacing (Cheng and Amin, 1983). The goal is to propose the best method of parameters estimation for Weibull distribution. We compare several methods for estimating the parameters of the two- and three-parameter Weibull distribution.

## Materials and Methods

### *Materials*

This study was conducted in a semi natural beech dominated forest in district 1, compartment 111, located at Kordkuy forest in Golestan Province, northern Iran (UTM zone 40: E247508, N4065346). The forest had been managed with single selection harvesting techniques until beginning of 2018. Elevation of the study area is about 1488 meters above sea level, with an average temperature of 15 degrees centigrade and a mean annual precipitation of 600 mm (G.N.R.W.M., 2017). The geological substratum is Precambrian sediments, mainly composed of metamorphic schist which is known as Gorgan green schist (Kurdi et al., 2017). The soil has been classified as a forest brown soil. One 100 × 100 m plot was established in an area in the compartment where the cover of the tree canopy projection was over 85%. All living trees with dbh greater than 7 cm within the plots were measured using two caliper readings at an angle of 90° to one another. The summary statistics are provided in Table 1. The study area consists of mixed deciduous and uneven-aged forests, dominated by *Fagus orientalis* Lipsky (91.19%) associated with *Carpinus betulus* (9.95%), *Acer velutinum* (1.55%), *Alnus subcordata* (0.51%), *Tilia begoniifolia* Steven (0.51%) and *Prunus avium* L. (0.25%) species.

**Table 1.** Estimation results for fitting three-parameter Weibull model to DBH data.

| Number of trees | Mean | St. Dev. | Min. | Max. | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| 386 | 25.62 | 20.73 | 7.6 | 101.50 | 1.68 | 2.18 |

### *Methods*

Let $x_1, x_2, \ldots, x_n$ denote the observed random sample from two-parameter Weibull distribution. Also, let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ denote the ordered random observations. In what follows, we review ten known methods for estimating the parameters of the two- (Subsections 2.2.1-2.2.10) and three-parameter (Subsections 2.2.11-2.2.17) Weibull distributions. All methods in two- and three-parameter cases have been implemented using the package called ForestFit (Teimouri et al., 2020) developed for R (R Core Team, 2018) environment and uploaded to CRAN (Comprehensive R Archive Network) at https://cran.r-project.org/ web/ packages/ ForestFit/index.html.

### *Maximum likelihood (ML)*

There is no closed-form expression for maximum likelihood estimator (MLE) of the two-parameter Weibull parameters. It is asymptotically normal and efficient for large sample sizes. Many attempts have been made to compute or modify the MLEs of the Weibull distribution parameters. The MLE of the shape parameter is computed as the root of the equation:

$$\frac{n}{\alpha} - \sum_{i=1}^{n} \log x_i - n \log \beta + \sum_{i=1}^{n} \left( \frac{x_i}{\beta} \right)^{\alpha} \log \left( \frac{x_i}{\beta} \right),$$

and the MLE of the scale parameter is given by (Norman et al., 1994):

$$\beta_{MLE} = \left( \frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{\alpha}_{MLE}} \right)^{\frac{1}{\hat{\alpha}_{MLE}}}.$$

It can be seen that $\beta_{MLE}$ depends on $\alpha_{MLE}$ that must be computed numerically.

***Weighted maximum likelihood (WML)***

It is known that MLEs are generally biased. To reduce the bias rate the weighted maximum likelihood estimators (WMLEs) have been proposed in the literature (Jacquelin, 1993). Then WMLEs of the shape and scale parameters are given by:

$$\hat{\alpha}_{WMLE} = \arg\min_{\alpha}\left(\frac{W_2}{\alpha}+\frac{1}{n}\log x_i-\frac{\sum_{i=1}^{n}x_i^{\alpha}\log x_i}{\sum_{i=1}^{n}x_i^{\alpha}}\right)^2,$$

$$\hat{\beta}_{WMLE} = \left(\frac{1}{nW_1}\sum_{i=1}^{n}x_i^{\alpha}\right)^{\frac{1}{\alpha}},$$

where the weights $W_1$ and $W_2$ are given by:

$$W_1 = -\frac{1}{n}\sum_{i=1}^{n}\log(1-F(X_i)),$$

$$W_2 = \frac{\sum_{i=1}^{n}\log(1-F(X_i))\log\left[-\log(1-F(X_i))\right]}{\sum_{i=1}^{n}\log(1-F(X_i))} -\frac{1}{n}\sum_{i=1}^{n}\log\left[-\log(1-F(X_i))\right].$$

Although the sampling distribution of the $W_1$ is gamma with shape parameter $n$ and scale parameter $1/n$, but the sampling distribution of the $W_2$ is not known. In practice, both random variables $W_1$ and $W_2$ are replaced by their median (Cousineau, 2009). Median of these random variables are computed by performing a Monte Carlo simulation for different levels of small sample size $n=1,\ldots,100$. When $n$ gets large, both WML and ML approaches give the same results.

***Generalized least square of first type (GLS1)***

The parameter estimation using least square approach is common in the statistical literature. We can see that the following regression model holds.

$$y_{(i)} = -\alpha\log\beta+\alpha\log[-\log(1-F(x_{(i)}))] \quad (3)$$

for $i=1,\ldots,n$ where $y_{(i)}=\log x_{(i)}$. The quantity $F(x_{(i)})$, in the right-hand side of regression model (Burkhart and Tomé, 2012), is replaced by $i/(n+1)$ or $(i-0.3)/(n+0.4)$ (Tiryakioğlu and Hudak, 2007; Van Zyl and Schall, 2012). Since the sample $x_{(i)}$ is ordered, the dependent variable $y_{(i)}$ is also *ordered*. Therefore the variance of dependent variable is not of the form $\sigma^2 \mathrm{I}$ (Kantar, 2015). To tackle this issue the generalized least square (GLS) technique is proposed (Engeman and Keefe, 1982). The GLS estimate, i.e., $\hat{\beta}_{GLS}=(-\hat{\alpha}\log\hat{\beta},\ \hat{\alpha})^T$ is given by:

$$\hat{\beta}_{GLS1} = (X^TV^{-1}X)X^TV^{-1}Y, \quad (4)$$

where $Y=\left(\log x_{(1)},\log x_{(2)},\ldots,\log x_{(n)}\right)^T$,

$$X=\begin{pmatrix}1 & \log\left[-\log(1-\hat{F}(x_{(1)}))\right]\\ \vdots & \vdots\\ 1 & \log\left[-\log(1-\hat{F}(x_{(n)}))\right]\end{pmatrix},$$

and

$$V=\begin{pmatrix}v_{11} & \cdots & v_{1n}\\ \vdots & \vdots & \vdots\\ v_{n1} & \cdots & v_{nn}\end{pmatrix},$$

for

$$v_{ij}=\frac{i}{(n+1-i)\left[\log(n+1-i)-\log(n+1)\right]\left[\log(n+1-j)-\log(n+1)\right]}, \quad (5)$$

where $i \le j$.

***Generalized least square of second type (GLS2)***

The second type of GLS estimate, i.e.,

$$\hat{\beta}_{GLS\,2} = (Z^T V^{-1} Z) Z^T V^{-1} Y\,, \qquad (6)$$

can be constructed if we replace $X$ with $Z$ which is defined as

$$Z = \begin{pmatrix} 1 & \log(-\log(1-\hat{F}(x_{(1)}))) - 0.5 - \dfrac{\log(1-\hat{F}(x_{(1)}))}{((1-\hat{F}(x_{(1)}))\log(1-\hat{F}(x_{(1)})))^2} \\ \vdots & \vdots \\ 1 & \log(-\log(1-\hat{F}(x_{(n)}))) - 0.5 - \dfrac{\log(1-\hat{F}(x_{(n)}))}{((1-\hat{F}(x_{(n)}))\log(1-\hat{F}(x_{(n)})))^2} \end{pmatrix},$$

in (4). It is worth noting that $\hat{\beta}_{GLS} = (-\hat{\alpha}\log\hat{\beta},\ \hat{\alpha})^T$ and $\hat{F}(x_{(i)}) = \dfrac{i}{n+1}$.

***Weighted least square (WLS)***

The weighted least square (WLS) estimate is also given by

$$\hat{\beta}_{WLS} = (X^T W^{-1} X) X^T W^{-1} Y\,, \qquad (7)$$

where $\hat{\beta}_{WLS} = (\log\hat{\beta}, 1/\hat{\alpha})^T$, $v_{ii}$; for $i = 1,\ldots,n$, are coming from (5), and $W$ is a diagonal matrix whose entries are $v_{11},\ldots,v_{nn}$ (Kantar, 2015).

***L-moment Method (LM)***

By equating the sample *L*-moment with the population counterpart, we can obtain the *L*-moment estimators (Hosking, 1990). The *r*-th *L*-moment of two-parameter Weibull distribution is given by:

$$\mu_r^L = \frac{\beta}{r}\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}(r-k)C_{r-k}^{r}\sum_{j=0}^{r-k-1}C_j^{r-k-1}\frac{(-1)^j}{(k+j+1)^{1/\alpha+1}},$$

where $\alpha > 0$, $\beta > 0$, $r = 1,2,\ldots$, and $C_i^n$ denotes the binomial coefficient $n!/(i!(n-i)!)$ (Hosking, 1990). So the first and the second L-moments are given by $\mu_1^L = \beta\Gamma(1/\alpha+1)$ and $\mu_2^L = \beta\Gamma(1/\alpha+1)(1-2^{-\frac{1}{\alpha}})$, respectively. The first two sample L-moments are:

$$m_1^L = \frac{1}{n}\sum_{i=1}^{n} X_{i:n} = \overline{X},$$

and

$$m_2^L = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)X_{i:n} - \overline{X}.$$

Now, equating $\mu_1^L$ and $\mu_2^L$ with $m_1^L$ and $m_2^L$, respectively, the *L*-moments of $\alpha$ and $\beta$ are obtained as:

$$\alpha_{LM} = -\frac{\ln(2)}{\ln\left(1 - m_2^L / m_1^L\right)},$$

and

$$\beta_{LM} = \frac{m_1^L}{\Gamma\left(1/\alpha_{LM}+1\right)}.$$

***Logarithmic moment method (MLM)***

The logarithmic moment estimators of the shape and scale parameters of two-parameter Weibull distribution are given by (Dodson, 2006; Wayne, 1982; Norman et al., 1994):

$$\alpha_{MLM} = \sqrt{\frac{\pi^2}{6S^2}}, \qquad (8)$$

and

$$\beta_{MLM} = \exp\{M_1 - \psi(1)/\alpha_{MLM}\}, \qquad (9)$$

where $S^2$ and $M_1$ are the sample variance and the mean of log-transformed data, respectively. Also $\psi(1) = -0.5772156$. It has been shown by Cousineau, (2009) and Dodson, (2006) that the estimator is both asymptotically unbiased and consistent (Norman et al., 1994).

***Percentile method (PM)***

The quantile of a two-parameter Weibull distribution is:

$$x_p = \beta\left[-\ln(1-p)\right]^{1/\alpha},$$

where $0 < p < 1$, (Dodson, 2006). Using $p = 1-\exp(-1) \cong 0.632$, one can construct percentile-based estimators for $\alpha$ and $\beta$ as:

$$\hat{\alpha}_{PM} = \left(\frac{\ln[-\ln(1-p)]}{\ln\left(x_p\right) - \ln\left(x_{0.632}\right)}\right), \tag{10}$$

and

$$\hat{\beta}_{PM} = x_{1-\exp(-1)}, \tag{11}$$

respectively, where $0 < x_p < x_{0.632}$. The suggested values for $p$ are 0.15 (Wang and Keats, 1995) and 0.31 (Hassanein, 1971). Statistical tools show that percentile-based estimators are, in general, asymptotically normal and unbiased (Wayne, 1982).

***Moment method (MM)***

The $r$-th non-central moment for the Weibull distribution is (Dodson, 2006; Norman et al., 1994; Wayne, 1982):

$$\mu_r = \beta^r \Gamma\left(r/\alpha + 1\right).$$

Equating the mean and variance ($\mu_1$ and $\mu_2 - \mu_1^2$) with the sample counterparts ($\overline{X}$ and $S^2$), the moment-based estimator of the shape parameter $\alpha_{MM}$, is root of the equation:

$$\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)} + \frac{S^2}{\overline{\overline{X}}} - 1 = 0,$$

and the moment-based estimator of the scale parameter is:

$$\beta_{MM} = \frac{\overline{X}}{\Gamma\left(1/\alpha_{MM} + 1\right)}.$$

***Method of rank (MR)***

The method of rank estimator for the shape parameter is given by (Teimouri and Nadarajah, 2012):

$$\hat{\alpha}_{MR} = \frac{\log(2)}{\log(1 - \rho \frac{CV}{\sqrt{3}}\sqrt{\frac{n+1}{n-1}})}$$

where $\rho$ denotes the sample correlation between the $x_i$ s and their ranks and $CV$ is the sample coefficient of variation. It was shown that $\alpha_{MR}$ performs as good as the MLE and so we estimate the scale parameter as:

$$\beta_{MR} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\alpha}_{MR}}\right)^{\frac{1}{\hat{\alpha}_{MR}}}.$$

In what follows, we review these methods to estimate parameters of three-parameter Weibull distribution.

***TL-moment (TL)***

The *TL*-moment, i.e., $\mu_r^t$ is defined as (Elamir and Seheult, 2003; Abdul-Moniem, 2007; Teimouri et al., 2013):

$$\mu_r^t = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} E\left(X_{r+t-k:r+2t}\right) \tag{12}$$

for $t, r = 1, 2, 3, \ldots$, where $C_i^n$ denotes the binomial coefficient $n!/(i!(n-i)!)$ and $E(X_{(i)})$ is the expectation of the i-th order statistic in a sample of size $n$. Note that L-moments correspond to taking $t = 0$ in (Engeman and Keefe, 1982). The sample *TL*-moments are defined as:

$$m_r^t = \frac{\sum_{i=t+1}^{n-t} \frac{\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} C_{t+k}^{n-i} C_{r+t-k-1}^{i-1}}{C_{r+2t}^{n}} X_{(i)}}{r}. \tag{13}$$

Consider the expectation of the $i^{th}$ order statistic (Teimouri et al., 2013) and substitute $t=1$ and $r=1,2$ (Engeman and Keefe, 1982). The estimators of the $\alpha$, $\theta$, and $\beta$ in terms of the *TL*-moments are obtained as:

$$\mu_1^1 = \frac{3\beta\Gamma(1/\alpha+1)}{2^{1/\alpha}} - \frac{2\beta\Gamma(1/\alpha+1)}{3^{1/\alpha}} + \theta, \quad (14)$$

and

$$\mu_2^1 = 6\beta\Gamma(1/\alpha+1)\sum_{j=0}^{2} C_j^2 \frac{(-1)^j}{(j+2)^{1/\alpha+1}} - 6\beta\Gamma(1/\alpha+1)\sum_{j=0}^{1}\frac{(-1)^j}{(j+3)^{1/\alpha+1}}, \quad (15)$$

where $\Gamma(u)=\int_0^\infty y^{u-1}e^{-y}dy$ ; for $u>0$. Correspondingly, the first two sample *TL*-moments are given from (13) by:

$$m_1^1 = \frac{6}{n(n-1)(n-2)}\sum_{i=2}^{n-1}(i-1)(n-i)X_{i:n} \quad (16)$$

and

$$m_2^1 = \frac{12}{n(n-1)(n-2)(n-3)}\left[\sum_{i=3}^{n-1} C_1^{n-i} C_2^{i-1} X_{i:n} - \sum_{i=2}^{n-2} C_2^{n-i} C_1^{i-1} X_{i:n}\right]. \quad (17)$$

Consider the first moment of three-parameter Weibull distribution:

$$\mu = \theta + \beta\Gamma(1/\alpha+1). \quad (18)$$

Subtracting (14) from (18) and dividing the result by (15) yields:

$$\frac{2^{-1/\alpha-1} - 3^{-1/\alpha-1} - 1}{\sum_{j=0}^{2} C_j^2 \frac{(-1)^j}{(j+2)^{1/\alpha+1}} - \sum_{j=0}^{1}\frac{(-1)^j}{(j+3)^{1/\alpha+1}}}. \quad (19)$$

Equate (19) to the sample counterpart, i.e., $(m_1^1-\bar{X})/m_2^1$ to obtain an estimator of $\alpha$ as:

$$\hat{\alpha}_{TL} = \left\{\alpha \,|\, \frac{2^{-1/\alpha-1} - 3^{-1/\alpha-1} - 1}{\sum_{j=0}^{2} C_j^2 \frac{(-1)^j}{(j+2)^{1/\alpha+1}} - \sum_{j=0}^{1}\frac{(-1)^j}{(j+3)^{1/\alpha+1}}} - \frac{m_1^1-\bar{X}}{m_2^1} = 0\right\}. \quad (20)$$

Once we have obtained $\hat{\alpha}_{TL}$, by equating (17) and (17), $\beta$ is estimated as:

$$\beta_{TL} = \frac{m_2^1}{6\Gamma(1/\hat{\alpha}_{TL}+1)}\left[\sum_{j=0}^{2}\frac{C_j^2(-1)^j}{(j+2)^{1/\hat{\alpha}_{TL}+1}} - \sum_{j=0}^{1}\frac{(-1)^j}{(j+3)^{1/\hat{\alpha}_{TL}+1}}\right]^{-1},$$

and similarly by equating (14) and (16), $\theta$ is estimated as:

$$\theta_{TL} = m_1^1 - \frac{3\beta\Gamma(1/\hat{\alpha}_{TL}+1)}{2^{1/\hat{\alpha}_{TL}}} + \frac{2\beta\Gamma(1/\hat{\alpha}_{TL}+1)}{3^{1/\hat{\alpha}_{TL}}}.$$

***Maximum product spacing (MPS)***

The method of MPS can be considered as an alternative to the maximum likelihood (ML) method for estimating the parameters of a continuous univariate distribution (Cheng and Amin, 1983). This approach was introduced for the shifted continuous univariate distributions when the ML estimators break down. The MPS estimators are asymptotically normal and as efficient (Cheng and Amin, 1979). The MPS approach works on the basis of: maximizing the mean of log-spacing function $S(\alpha,\beta,\theta)$ defined as

$$S(\alpha,\beta,\theta)=\frac{1}{m}\sum_{i=1}^{m}\log\left[F(x_{(i)},\alpha,\beta,\theta)-F(x_{(i-1)},\alpha,\beta,\theta)\right]$$
$$=\frac{1}{m}\sum_{i=1}^{m}\log\left[\exp\left\{-\left(\frac{x_{(i-1)}-\theta}{\beta}\right)^{\alpha}\right\}-\exp\left\{-\left(\frac{x_{(i)}-\theta}{\beta}\right)^{\alpha}\right\}\right], \quad (21)$$

with respect to $(\alpha,\beta,\theta)^T$ where $F(x_{(0)},\alpha,\beta,\theta)=0$ and $F(x_{(m)},\alpha,\beta,\theta)=1$ with $m=n+1$. Here, in order to compute the MPS estimators of three-parameter Weibull distribution, we use the *optim* function developed for R environment to maximize the right-hand side of (21).

***Modified maximum likelihood (MML)***

Based on MML approach, the log-likelihood function of a three-parameter Weibull distribution is given by Cohen and Whitten, (1982) as:

$$l(\alpha,\beta,\theta)=n\log(\alpha)-n\alpha\log(\beta)-(\alpha-1)\sum_{i=1}^{n}\log(x_i-\theta)-\sum_{i=1}^{n}\left(\frac{x_i-\theta}{\beta}\right)^{\alpha}, \quad (22)$$

is maximized with respect to $(\alpha,\beta,\theta)^T$ under three scenarios. In the following we review three types of the MML briefly.

• **MML type I:** The log-likelihood function (22) is maximized subject to the constraint:

$$\log\left(\frac{n}{n+1}\right)=\left(\frac{x_{(1)}-\theta}{\beta}\right)^{\alpha}. \quad (23)$$

The first ordered observation is sufficient statistic for $\theta$. So, the right-hand side of (22) is maximized with respect to $(\alpha,\beta,\theta)^T$ when $\log\left(n/(n+1)\right)$ is replaced with $(x_{(1)}-\theta)^{\alpha}/\beta^{\alpha}$.

• **MML type II:** In this case, the $l(\alpha,\beta,\theta)$ in (22) is maximized subject to the constraint:

$$E(X_{(1)})=\theta+\beta n^{-\frac{1}{\alpha}}\Gamma(1/\alpha+1). \quad (24)$$

So, the $l(\alpha,\beta,\theta)$ is maximized with respect to $(\alpha,\beta,\theta)^T$ when $x_{(1)}$ is replaced with right-hand side of (24).

• **MML type III:** It is well known that $E(X)=\bar{X}$ and $E(X)=\theta+\beta\Gamma(1/\alpha+1)$. The MML type III is obtained by maximizing the $l(\alpha,\beta,\theta)$ subject to the constraint:

$$\bar{X}=\theta+\beta\Gamma(1/\alpha+1). \quad (25)$$

• **MML type IV:** Taking into account the fact that the median of random variable $X$ is $\theta+\beta(\log(2))^{1/\alpha}$, the MML type IV is obtained by maximizing the $l(\alpha,\beta,\theta)$ subject to the condition:

$$X_m=\theta+\beta(\log(2))^{1/\alpha}, \quad (26)$$

where $X_m$ is the sample median.

***Method of moment (MM)***

The MM approach works by equating the first three population moments to the corresponding sample moments. Let $\bar{X}$, $S^2$, and $S^3$ denote the sample mean, variance, and skewness, respectively. It follows that (Dodson, 2006; Wayne, 1982; Norman et al., 1994):

$$S^3=\frac{\Gamma(3/\alpha+1)-2\Gamma(2/\alpha+1)\Gamma(1/\alpha+1)+2\Gamma^3(1/\alpha+1)}{\left[\Gamma(2/\alpha+1)-\Gamma^2(1/\alpha+1)\right]^{3/2}}, \quad (27)$$

$$S^2 = \beta^2[\Gamma(2/\alpha+1)-\Gamma^2(1/\alpha+1)], \quad (28)$$

$$\bar{X} = \theta+\beta\Gamma(1/\alpha+1). \quad (29)$$

After extracting $\alpha$ from equation (27) and substituting the extracted $\alpha$ into equations (28) and (29), the scale and location parameters are estimated by solving the equations (28) and (29), respectively.

***Method of modified moment (MMM)***

Similar to MML, the MM also uses the constraints to estimate the parameters (Cohen and Whitten, 1982).

• **MMM type I:** The shape parameter $\alpha$ is estimated by solving the equation given by:

$$\frac{S^2}{(\bar{X}-X_{(1)})^2} = \frac{\Gamma(2/\alpha+1)-\Gamma^2(1/\alpha+1)}{\left[\Gamma(1/\alpha+1)-[-\log(n/(n+1))]^{1/\alpha}\right]^2}.$$

Once we have obtained $\alpha$, as $\hat{\alpha}$, then $\beta$ is obtained as $\tilde{\beta}$ from equation:

$$S^2 = \beta^2\Gamma(2/\hat{\alpha}+1)-\beta^2\Gamma^2(1/\hat{\alpha}+1),$$

and then $\theta$ is estimated as:

$$\hat{\theta} = \bar{X}-\hat{\beta}\Gamma(1/\hat{\alpha}+1).$$

• **MMM type II:** The shape parameter $\alpha$ is estimated through solving the equation:

$$\frac{S^2}{(\bar{X}-X_{(1)})^2} = \frac{\Gamma(2/\alpha+1)-\Gamma^2(1/\alpha+1)}{[\Gamma(1/\alpha+1)(1-n^{-1/\alpha})]^2}.$$

Once we have obtained $\alpha$ as $\hat{\alpha}$, again $\beta$ is obtained as $\tilde{\beta}$ from equation:

$$S^2 = \beta^2\Gamma(2/\hat{\alpha}+1)-\beta^2\Gamma^2(1/\hat{\alpha}+1),$$

and then $\theta$ is estimated as:

$$\hat{\theta} = \bar{X}-\hat{\beta}\Gamma(1/\hat{\alpha}+1).$$

• **MMM type III:** The shape parameter $\alpha$ is estimated through solving the equation:

$$\frac{S^2}{(\bar{X}-X_{m})^2} = \frac{\Gamma(2/\alpha+1)-\Gamma^2(1/\alpha+1)}{[\Gamma(1/\alpha+1)-(\log(2))^{1/\alpha}]^2}.$$

Once we have obtained $\alpha$, as $\hat{\alpha}$, again $\beta$ is obtained as $\tilde{\beta}$ from equation:

$$S^2 = \beta^2\Gamma(2/\hat{\alpha}+1)-\beta^2\Gamma^2(1/\hat{\alpha}+1),$$

and then $\theta$ is estimated as:

$$\hat{\theta} = \bar{X}-\hat{\beta}\Gamma(1/\hat{\alpha}+1).$$

***Method of maximum likelihood (ML)***

The ML estimators of the parameters $\alpha$ and $\theta$ are given by:

$$\arg\min_{\alpha,\theta}\left\{\begin{array}{l}\left(\frac{1}{\alpha}+\frac{1}{n}\sum_{i=1}^{n}\log(x_i-\theta)-\frac{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}\log(x_i-\theta)}{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}}\right)^2+\\ \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(x_i-\theta)}\times\frac{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}}{\sum_{i=1}^{n}(x_i-\theta)^{\alpha-1}}-\frac{\alpha}{\alpha-1}\right)^2\end{array}\right\}.$$

After estimating $\alpha$ and $\theta$, the parameter $\beta$ will be estimated as:

$$\beta_{ML} = (\frac{1}{n}\sum_{i=1}^{n}(x_i-\theta)^{\alpha})^{1/\alpha}.$$

***Weighted maximum likelihood (WML)***

The WML estimators of the parameters

$\alpha$ and $\theta$ are given by:

$$\arg\min_{\alpha,\theta}\left\{\begin{array}{l}\left(\frac{W_2}{\alpha}+\frac{1}{n}\sum_{i=1}^{n}\log(x_i-\theta)-\frac{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}\log(x_i-\theta)}{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}}\right)^2+\\ \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(x_i-\theta)}\times\frac{\sum_{i=1}^{n}(x_i-\theta)^{\alpha}}{\sum_{i=1}^{n}(x_i-\theta)^{\alpha-1}}-W_3\right)^2\end{array}\right\},$$

where the weights $W_1$ and $W_2$ have been defined in Subsection 2.2.2, and $W_3$ is given by:

$$W_3=W_1\frac{\sum_{i=1}^{n}\left[\log(1-F(X_i))\right]^{1/\alpha}}{\sum_{i=1}^{n}\left[\log(1-F(X_i))\right]^{1-1/\alpha}}.$$

After estimating $\alpha$ and $\theta$, the parameter $\beta$ will be estimated as:

$$\beta_{WML}=\left[\frac{1}{nW_1}\sum_{i=1}^{n}(x_i-\theta)^{\alpha}\right]^{1/\alpha}.$$

The sampling distributions of the $W_3$ is unknown and is replaced by its median (Cousineau, 2009). For computing the median of $W_3$, we carried out a comprehensive Monte Carlo simulation for different levels of $\alpha$ (from 0.5 to 5 by 0.2) and small sample size $n=1,\ldots,100$.

***Goodness-of-fit test***

Here, we perform a real data analysis involving comparison between performances of all methods introduced in the previous subsection for estimating the parameters of two-and three-parameter Weibull distributions. These methods are applied to the dbh data described in previous subsection. To implement these techniques, programs have been written in R environment (R Core Team, 2018). In order to compare the performance of estimators we employed the Kolmogorov-Smirnov (KS) and Cramer-Von Mises (CVM) distances which are given by:

$$\text{KS}=\max_{1\le i\le n}\max\left\{\frac{i}{n}-F\left(x_{(i)};\hat{\alpha},\hat{\beta},\hat{\theta}\right),F\left(x_{(i)};\hat{\alpha},\hat{\beta},\hat{\theta}\right)-\frac{i-1}{n}\right\}$$

and

$$\text{CVM}=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-F(x_{(i)};\hat{\alpha},\hat{\beta},\hat{\theta})\right]^2,$$

where $F(.;\hat{\alpha},\hat{\beta},\hat{\theta})$ is the distribution function of two- or three-parameter Weibull distribution under estimated parameters $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\theta}$. We note that if a two-parameter Weibull distribution is fitted to the data, then $\hat{\theta}=0$.

## Results and Discussion

The results are given in Table 2 and Table 3. The histogram of data dbh is described in Subsection 2.1 superimposed by the fitted pdfs of two- and three-parameter Weibull given in Figure 1 and Figure 2, respectively.

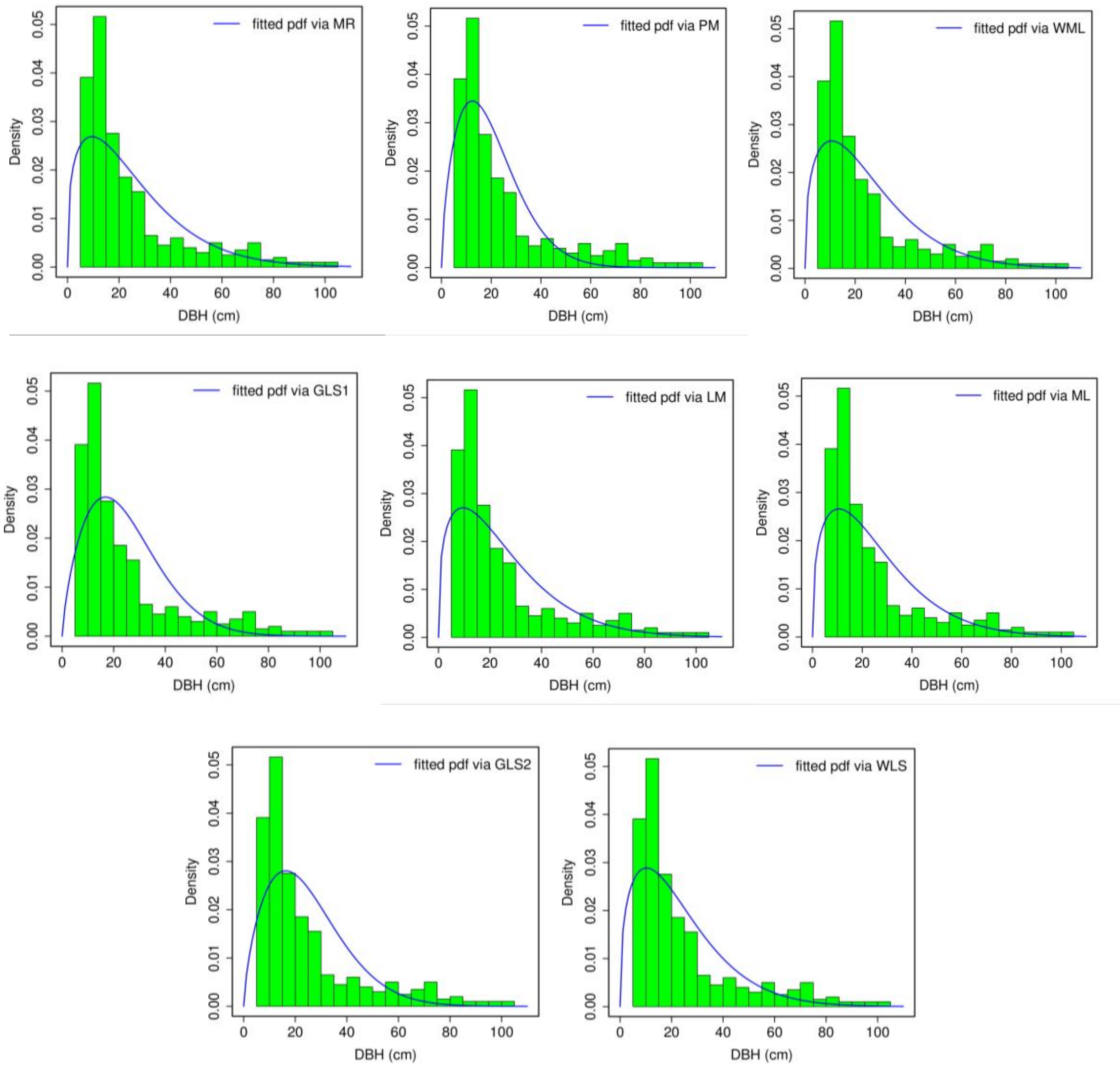


**Figure 1.** Fitted pdf to the dbh data using estimators for two-parameter Weibull distribution.

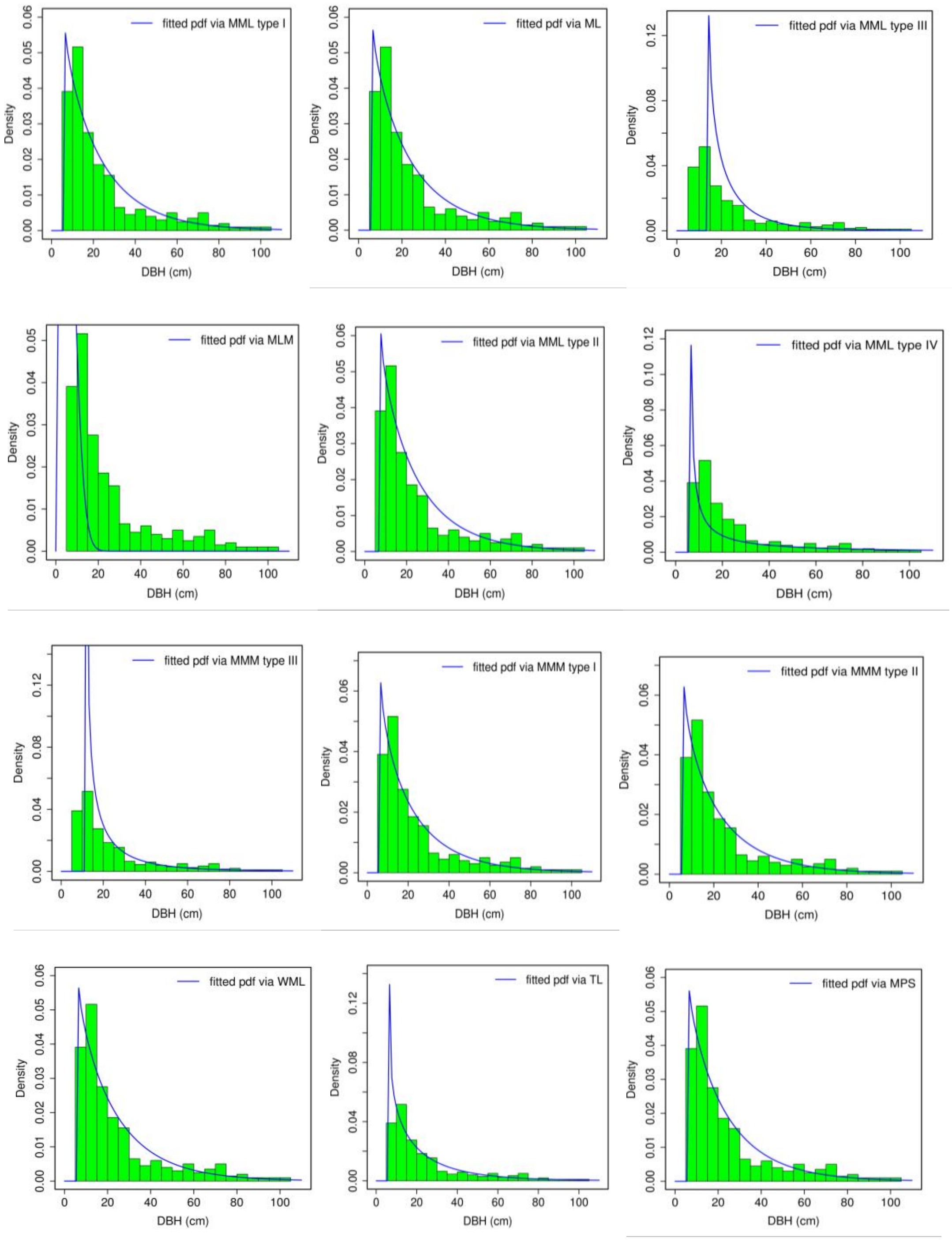


**Figure 2.** Fitted pdf to the dbh data using estimators for three-parameter Weibull distribution.

**Table 2.** Estimation results for fitting three-parameter Weibull model to dbh data.

| Method | Estimated parameters | | | goodness-of-fit measures | |
|---|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | KS | CVM |
| TL | 0.83508 | 16.76002 | 6.55764 | 0.06944 | 0.36494 |
| MPS | 0.96126 | 18.80060 | 5.96106 | 0.07018 | 0.60681 |
| MML type I | 0.96883 | 18.47404 | 6.00000 | 0.06774 | 0.54743 |
| MML type II | 0.96299 | 18.64702 | 5.63625 | 0.07054 | 0.56176 |
| MML type III | 0.38531 | 2.974900 | 14.0000 | 0.39849 | 10.6387 |
| MML type IV | 0.36690 | 28.51173 | 6.00000 | 0.25524 | 9.57417 |
| MM | 0.88827 | 10.35778 | 14.0208 | 0.39849 | 14.5437 |
| MMM type I | 0.92131 | 18.30872 | 5.97289 | 0.06385 | 0.49649 |
| MMM type II | 0.92136 | 18.31031 | 5.97180 | 0.06390 | 0.49681 |
| MMM type III | 0.65182 | 9.54890 | 11.99281 | 0.30161 | 5.01207 |
| ML | 0.94051 | 18.94471 | 6.00000 | 0.07092 | 0.62513 |
| WML | 0.94051 | 18.94471 | 6.00000 | 0.07092 | 0.62513 |

**Table 3.** Estimation results for fitting two-parameter Weibull model to dbh data.

| Method | Estimated parameters | | goodness-of-fit measures | |
|---|---|---|---|---|
| | $\hat{\alpha}$ | $\hat{\beta}$ | KS | CVM |
| MLE | 1.36620 | 27.61208 | 0.124644 | 2.35193 |
| WMLE | 1.36620 | 27.61208 | 0.12464 | 2.35193 |
| GLS1 | 1.72611 | 27.74422 | 0.17138 | 3.94360 |
| GLS2 | 1.68685 | 27.74996 | 0.16591 | 3.70796 |
| WLS | 1.39034 | 25.46912 | 0.13541 | 1.67160 |
| LM | 1.32747 | 27.17786 | 0.13471 | 2.14956 |
| MLM | 1.84778 | 6.763329 | 0.70576 | 97.0987 |
| PM | 1.62721 | 22.20840 | 0.12454 | 1.49667 |
| MM | 1.21517 | 26.66050 | 0.16118 | 2.17345 |
| MR | 1.32520 | 27.31969 | 0.13418 | 2.19901 |

The following observations can be made from Tables 2 and 3.

1. In two-parameter case, the percentile method outperforms other competitors in terms of both criteria KS and CVM.
2. In three-parameter case, the *TL*-moment method outperforms all other competitors for CVM criterion.
3. In three-parameter case, the modified method of moments type I and type II outperform the other competitors in terms of KS criterion.

Although the data used in this study come from an uneven-aged beech forest, the performance of method of moments in estimating parameters for three-parameter Weibull distribution is in agreement with the results reported by Lei (2008) in a pine plantation.

4. It should be noted that both ML and WML methods give the same estimators. This is due to the fact that all three weights corresponding to the WML method are computed just for samples of size less than or equal to 100. For sample sizes larger

than 100, similar to our real example, both ML and WML methods give the same estimators.

## Conclusion

We have compared the performance of the ten methods of estimating the two-parameter and twelve methods for estimating the three-parameter Weibull distributions for modelling the diameter at breast height of trees. The comparisons were made using Kolmogorov-Smirnov (KS) and Cramer Von-Mises (CVM) criteria. Although the one-hectare studied plot is a good representative of uneven-aged beech forest with a wide range of diameter distribution, it should be noted that changes in stand attributes may influence the results in other uneven-aged beech forest. Hence, caution should be emphasized before recommendations are made as to which methods are appropriate for parameter estimating of Weibull distribution. We note that although the data used in this study come from an uneven-aged beech forest, but the performance of method of moments in estimating the parameters of three-parameter Weibull distribution is in agreement with the results reported in the literature for pine population.

**Acknowledgements** The authors would like to thank two anonymous referees and the Executive Assistant, Dr. R. Safari, for their constructive comments that greatly improved the quality of paper.

## References

Abdul-Moniem, I.B. 2007. L-moments and TL-moments estimation for the exponential distribution. Far East Journal of Theoretical Statistics, 23, 1. 51.

Bailey, R.L., and Dell, T.R. 1973. Quantifying diameter distributions with the Weibull function. Forest Science, 19 (2), 97-104.

Burkhart, H.E., and Tomé, M. 2012. Modeling forest trees and stands. Springer Science & Business Media.

Cheng, R.C.H., and Amin, N.A.K. 1979. Maximum product of spacings estimation with application to the lognormal distribution. Math Report, 791p.

Cheng, R.C.H., and Amin, N.A.K. 1983. Estimating parameters in continuous univariate distributions with a shifted origin. Journal of the Royal Statistical Society. Series B (Methodological), pp. 394-403.

Cohen, C.A., and Whitten, B. 1982. Modified maximum likelihood and modified moment estimators for the three-parameter Weibull distribution. Communications in Statistics-Theory and Methods, 11 (23), 2631-2656.

Coomes, D.A., and Allen, R.B. 2007. Mortality and tree-size distributions in natural mixed-age forests. Journal of Ecology, 95 (1), 27-40.

Cousineau, D. 2009. Nearly unbiased estimators for the three-parameter Weibull distribution with greater efficiency than the iterative likelihood method. British Journal of Mathematical and Statistical Psychology, 62, 167-191.

Dodson, B. 2006. The Weibull Analysis Handbook. ASQ Quality Press.

Duan, A.G., Zhang, J.G., Zhang, X.Q., and He, C.Y. 2013. Stand diameter distribution modelling and prediction based on Richards function. PloS one, 8(4), e62605.

Elamir, E.A.H., and Seheult, A.H. 2003. Trimmed L-moments. Computational Statistics & Data Analysis, 43 (3), 299-314.

Engeman, R.M., and Keefe, T.J. 1982. On generalized least squares estimation of the Weibull distribution. Communications in Statistics-Theory and Methods, 11 (19), 2181-2193.

Ghalandarayeshi, S., 2019. Modeling diameter distribution of Fagus orientals Lipsky using probability density functions (Case study in Kordkoy-Golestan province). In, The 2nd National Conference on Natural Resources Management. Gonbad Kavous Univerity, pp. 1-6.

Ghalandarayeshi, Sh., Nord-Larsen, T., Kvist Johannsen, V., and Bo Larsen, J. 2017. Spatial patterns of tree species in Suserup Skov-a semi-natural forest in Denmark. Forest Ecology and Management, 406, 391-401.

G.N.R.W.M. 2017. Forest management plan. Technical report, Golestan Natural Resources and Watershed Management.

Gorgoso, J.J., Álvarez González, J.G., Rojo, A., and Grandas-Arias, J.A. 2007. Modelling diameter distributions of Betula alba L. stands in northwest Spain with the two-parameter Weibull function. Forest Systems, 16 (2), 113-123.

Hassanein, Kh.M. 1971. Percentile estimators for the parameters of the Weibull distribution. Biometrika, 58 (3), 673-676.

Hosking, J.R.M. 1990. L-moments: analysis and estimation of distributions using linear combinations of order statistics. Journal of the Royal Statistical Society. Series B (Methodological), pp. 105-124.

Hung, W.L. 2001. Weighted least-squares estimation of the shape parameter of the Weibull distribution. Quality and Reliability Engineering International, 17 (6), 467-469.

Hyink, D.M., and Moser, J.W. 1983. A generalized framework for projecting forest yield and stand structure using diameter distributions. Forest Science, 29 (1), 85-95.

Jacquelin, J. 1993. Generalization of the method of maximum likelihood. IEEE Transactions on Electrical Insulation, 28 (1), 65-72.

Kantar, Y.M. 2015. Generalized least squares and weighted least squares estimation methods for distributional parameters. REVSTAT-Statistical Journal, 13 (3), 263-282.

Kilkki, P., Maltamo, M., Mykkänen, R., and Päivinen, R. 1989. Use of the Weibull function in estimating the basal area dbh-distribution. Silva Fennica, 23 (4), 311-318.

Kurdi, M., Soltani-Mohammadi, S., Eslamkish, T., and Larina, N. 2017. Evaluating the performance of concentration-number (cn) fractal model for separation of soil horizon regarding vertical distribution and 3d models. Journal of Tethys, 5 (4), 337-349.

Lei, Y. 2008. Evaluation of three methods for estimating the Weibull distribution parameters of Chinese pine (Pinus tabulaeformis). Journal of Forest Science, 54 (12), 566-571.

Little, S.N. 1983. Weibull diameter distributions for mixed stands of western conifers. Canadian Journal of Forest Research, 13 (1), 85-88.

Merganič, J., and Sterba, H. 2006. Characterisation of diameter distribution using the Weibull function: method of moments. European Journal of Forest Research, 125 (4), 427-439.

Newton, P.F., Lei, Y., and Zhang, S.Y. 2005. Stand-level diameter distribution yield model for black spruce plantations. Forest Ecology and Management, 209 (3), 181-192.

Norman, L.J., Kotz, S., and Balakrishnan, N. 1994. Continuous Univariate Distributions, volume I. John Wiley & Sons.

Prabhakar Murthy, D.N., Xie, M., and Jiang, R. 2004. Weibull Models, volume 505. John Wiley & Sons.

R Core Team. 2018. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. Available from URL:https://www.R-project.org/.

Reynolds, M.R., Burk, T.E., and Huang. W.Ch. 1988. Goodness-of-fit tests and model selection procedures for diameter distribution models. Forest Science, 34 (2), 373-399.

Stankova, T.V., and Zlatanov, T.M. 2010. Modeling diameter distribution of Austrian black pine (*Pinus nigra* Arn.) plantations: a comparison of the Weibull frequency distribution function and percentile-based projection methods. European Journal of Forest Research, 129 (6), 1169-1179.

Teimouri, M., and Nadarajah, S. 2012. A simple estimator for Weibull shape parameter. International Journal of Structural Stability and Dynamics, 12 (2), 395-402.

Teimouri, M., Hoseini, S.M., and Nadarajah, S. 2013. Comparison of estimation methods for the Weibull distribution. Statistics, 47 (1), 93-109.

Teimouri, M., Doser, J.W., and Finley, A.O. 2020. ForestFit: An R package for modeling plant size distributions, Environmental Modelling & Software, 131: 104668, doi:10.1016/j.envsoft.2020.104668.

Tiryakioğlu, M., and Hudak, D. 2007. On estimating Weibull modulus by the linear regression method. Journal of Materials Science, 42 (24), 10173-10179.

Van Zyl, J.M., and Schall, R. 2012. Parameter estimation through weighted least-squares rank

regression with specific reference to the Weibull and Gumbel distributions. Communications in Statistics-Simulation and Computation, 41 (9), 1654-1666.
Wang, F.K., and Keats, J.B. 1995. Improved percentile estimation for the two-parameter Weibull distribution. Microelectronics Reliability, 35 (6), 883-892.
Wayne, N. 1982. Applied Life Data Analysis. John Wiley & Sons, New York.
Zhang, L., Packard, K.C., and Liu, Ch. 2003. A comparison of estimation methods for fitting Weibull and Johnson's SB distributions to mixed spruce fir stands in northeastern North America. Canadian Journal of Forest Research, 33 (7), 1340-1347.
Zhang, L., and Liu, Ch. 2006. Fitting irregular diameter distributions of forest stands by Weibull, modified Weibull, and mixture Weibull models. Journal of Forest Research, 11 (5), 369-372.
Zhang, L.F., Xie, M., and Tang, L.C. 2008. On weighted least squares estimation for the parameters of Weibull distribution. In Recent Advances in Reliability and Quality in Design, Pp: 57-84. Springer.
Zhang, X., Lei, Y., and Cao, Q.V. 2010. Compatibility of stand basal area predictions based on forecast combination. Forest Science, 56 (6), 552-557.